MODEL QUESTION PAPER OF MATHEMATICS OF SEMESTER –I

ALGEBRA (MJ-1)

Full Marks:-75

Pass Marks:-30

GROUP-A (Compulsory)

1.

- **a)** Express $-i$ in polar form. $1\times5=5$
- **b)** Define invertible function.
- **c)** Define row rank of a matrix.
- **d)** What do you mean by trivial solution of a system of linear homogeneous equations?
- **e)** Define ordered basis.

2. Solve- $x^4+x^2+1=0$. (5)

3. If $a\equiv b\,(\bmod n)$ then show that $a^k\equiv b^k\,(\bmod n)$ where a,b,c,n and k and positive integers. (5)

GROUP-B (Answer any four questions) $15\times4=60$

4. State and prove De-Moivre's theorem.

5. Show that the function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=\dfrac{x^4+4x+30}{x^2-8x+18}$ is not one-one.

6. State and prove Fundamental theorem of arithmetic.

7. Show that elementary transformations do not change the rank.

8. Solve the system: $5x+3y+7z=4$

$$3x+26y+2z=9$$

$$7x+2y+10z=5$$

9. Determine the Eigen values and Eigen vectors of the matrix:

$$A=\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}.$$

10. Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be the operator defined by $T(x,y)=(2x+3y,4x-5y)$ then

i. Show that T is linear.
ii. Find matrix representation of T relative to usual basis of $\mathbb{R}^2$.
iii. Find the matrix representation of T relative to basis $\{(1,2),(2,5)\}$.

HINTS AND SOLUTIONS

1(a) $\cos\left(\frac{\pi}{2}\right)-i\sin\left(\frac{\pi}{2}\right)$.

(b) A function f is said to be invertible iff f is one-one and onto.

(c) Number of linearly independent rows in a matrix is called its row rank.

(d) Zero solution is called trivial solution.

(e) A basis endowed with a specific order.

2. Given: $x^4+x^2+1=0$

$$\Rightarrow \left(x^2-1\right)\left(x^4+x^2+1\right)=0$$

$$\Rightarrow \left(x^6-1\right)=0$$

$$\Rightarrow x=(1)^{\frac{1}{6}}$$

$$=\cos\left(\frac{2r\pi}{6}\right)+i\sin\left(\frac{2r\pi}{6}\right);\ r=0,1,2,3,4,5$$

$$=\cos 0+i\sin 0\ ,\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right),\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right),\cos\pi+i\sin\pi\ ,$$

$$\cos\left(\frac{4\pi}{3}\right)+i\sin\left(\frac{4\pi}{3}\right),\cos\left(\frac{5\pi}{3}\right)+i\sin\left(\frac{5\pi}{3}\right)$$

$$=\pm 1,\pm\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$$

$\Rightarrow$Solutions of $x^4+x^2+1=0$ are $\pm\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$.(Since ± 1 are solutions of $\left(x^2-1\right)$ and $\left(x^2-1\right)$ is not a factor of given equation).

3. $a\equiv b\left(\operatorname{mod} n\right)$

$$\Rightarrow n/a-b$$

$$\Rightarrow n/a-b\left(a^{k-1}+a^{k-2}b+a^{k-3}b^2+..........+b^{k-1}\right)$$

$\Rightarrow n/a^k - b^k \qquad \left(\because a^k - b^k = (a-b)\left(a^{k-1} + a^{k-2}b + a^{k-3}b^2 + ... + a^{k-2}b + b^{k-1}\right)\right)$

$\Rightarrow a^k \equiv b^k \pmod n$.

4. Statement:

i. n is a positive integer, then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

ii. If n is a rational number, then one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$.

Proof: CASE I- When n is a positive integer.

Consider $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$$= (\cos\theta_1 \cdot \cos\theta_2 - \sin\theta_1 \cdot \sin\theta_2) + i(\cos\theta_1 \cdot \sin\theta_2 + \sin\theta_1 \cdot \cos\theta_2)$$

$$= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$$

Again, $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)$

$$= \left((\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\right)(\cos\theta_3 + i\sin\theta_3)$$

$$= \left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right)\cos\theta_3 + i\sin\theta_3 \qquad \text{[From previous result]}$$

$$= \cos(\theta_1 + \theta_2 + \theta_3) + i\sin(\theta_1 + \theta_2 + \theta_3)$$

By similar process, we get

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)..........(\cos\theta_n + i\sin\theta_n)$$

$$= \cos(\theta_1 + \theta_2 + ... + \theta_n) + i\sin(\theta_1 + \theta_2 + ... + \theta_n)$$

Put $\theta = \theta_1 = \theta_2 = ... = \theta_n$, we get

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

This proves in this case.

CASE II- When n is a negative integer.

Let $n = -m$.

then $(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$

$$= \frac{1}{(\cos\theta + i\sin\theta)^m}$$

$$= \frac{1}{(\cos m\theta + i\sin m\theta)} \qquad \text{[From Case I]}$$

$$= \frac{(\cos m\theta - i\sin m\theta)(\cos m\theta + i\sin m\theta)}{(\cos m\theta + i\sin m\theta)}$$

$$= \frac{(\cos m\theta - i\sin m\theta)}{(\cos^2 m\theta - i^2 \sin^2 m\theta)}$$

$$= \frac{(\cos m\theta - i\sin m\theta)}{(\cos^2 m\theta + \sin^2 m\theta)}$$

$$= \frac{(\cos m\theta - i\sin m\theta)}{1}$$

$$= \cos(-m)\theta + i\sin(-m)\theta$$

$$= \cos n\theta + i\sin n\theta$$

CASE III- Let $n = \frac{p}{q}$ is a negative integer and p is any integer.

Since $\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^q = \cos q\frac{\theta}{q} + i\sin q\frac{\theta}{q}$

$$= \cos\theta + i\sin\theta \qquad \text{[From Case I]}$$

$\Rightarrow$One value of $(\cos\theta + i\sin\theta)^{\frac{1}{q}}$ is $\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}$.

$\Rightarrow$ One value of $(\cos\theta + i\sin\theta)^{\frac{p}{q}} = \left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^{p} = \cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta$.

$$= \cos n\theta + i\sin n\theta.$$

This proves De-Moivre's theorem.

5. Let $f(x_1) = f(x_2)$

$$\Rightarrow \frac{x_1^2 + 4x_1 + 30}{x_1^2 - 8x_1 + 18} = \frac{x_2^2 + 4x_2 + 30}{x_2^2 - 8x_2 + 18}$$

$$\Rightarrow \left(x_1^2 + 4x_1 + 30\right)\left(x_2^2 - 8x_2 + 18\right) = \left(x_2^2 + 4x_2 + 30\right)\left(x_1^2 - 8x_1 + 18\right)$$

$$\Rightarrow x_1^2x_2^2 - 8x_1^2x_2 + 18x_1^2 + 4x_1x_2^2 - 32x_1x_2 + 72x_1 + 30x_2^2 - 240x_2 + 540$$

$$= x_1^2x_2^2 - 8x_1x_2^2 + 18x_2^2 + 4x_1^2x_2 - 32x_1x_2 + 72x_2 + 30x_1^2 - 240x_1 + 540$$

$$\Rightarrow 4x_1x_2(x_1 - x_2) + 30(x_1 - x_2)(x_1 + x_2) + 8x_1x_2(x_1 - x_2)$$
$$-240(x_1 - x_2) - 8\left(x_1^2 - x_2^2\right) - 72(x_1 - x_2) = 0$$

$\Rightarrow 12x_1x_2 + 12(x_1 + x_2) - 312 = 0$ [If $x_1 \neq x_2$]

$\Rightarrow x_1x_2 + x_1 + x_2 = 26$ .... (1)

Take $x_1 = 0$, $x_2 = 26$, we see this satisfies equation(1)

$\Rightarrow f(0) = f(26)$ but $0 \neq 26$.

Hence, f is not one-one.

6. **<u>Statement</u>:** Every positive integer can be expressed as product of primes and it is unique apart from order of prime factors.

<u>Proof</u>: Let $n > 1$ be an integer. If n is prime then $n = n$ and if n is not prime then there exist a prime p_1 such that p_1/n.

$\Rightarrow n = p_1 n_1$ for some $n_1 \in \mathbb{Z}$

Again, if n_1 is prime then $n = p_1 n_1$

and if n_1 is not prime then there exist a prime p_2 such that p_2 / n_1.

$\Rightarrow n_1 = p_2 n_2$

$\Rightarrow n = p_1 p_2 n_2$

If n_2 is prime then we are done and if not then continue the above process.

Since there are only finite number of prime factors of n, therefore after finite number of steps, we get

$n = p_1 p_2 .... p_r$; where each p_i is a prime number.

Hence n can be expressed as product of primes.

Uniqueness:

Let $n = p_1 p_2 .... p_r$

and $n = q_1 q_2 .... q_s$; where p_i's and q_i's are primes.

If possible, let $r < s$.

$\because p_1 / n$

$\therefore p_1 / q_1 q_2 .... q_s$

$\Rightarrow p_1 / q_j$ for some $j = 1, 2, 3, ..., s$

Let p_1 / q_1

$\Rightarrow p_1 = q_1$ [$\because$ both are primes]

In the similar way, we can show

$p_2 = q_2$
$p_3 = q_3$
....
....
....
$p_r = q_r$

$\because p_1 p_2 .... p_r = q_1 q_2 .... q_r q_{r+1} q_{r+2} .... q_s$

$\therefore 1 = q_{r+1} q_{r+2} .... q_s$, which is a contradiction.

$\Rightarrow r = s$ and $p_i = q_j$ for some i and j.

Hence, the result.

7. Proof: Let $A = \left[a_{ij}\right]$ be an $m \times n$ matrix of rank r.

Now consider the following three stages one by one-

a) Interchanging of a pair of rows: Let B be the matrix obtained from A by interchanging its u^{th} and v^{th} rows. Let B_0 be any square sub-matrix of B of order $(r+1)$.

Then $|B_0| = 0$, if B_0 do not contains u^{th} and v^{th} rows of A [$\because$ in this case B_0 is also a sub-matrix of A].

Again, if B_0 contains both the rows u and v then $|B_0| = -|A_0| = 0$, where A_0 is corresponding sub-matrix of A in which u^{th} and v^{th} rows are in their own position.

Next, if B_0 contains one of u^{th} or v^{th} row say u^{th} row only then $|B_0| = 0$ [$\because$ row of B_0 are rows of A].

Hence, in all the cases we see that any $(r+1)$ order minor of B is zero.

Therefore, $\rho(B) \le r = \rho(A)$

Since by reversing the interchanging process of rows, we can obtain A from B.

$\therefore \rho(A) \le \rho(B)$

Thus, $\rho(A) = \rho(B)$.

b) Multiplication of an element in a row: Let B is obtained from A by the elementary transformation $R_i \to kR_i \ (k \ne 0)$. Let B_0 be any sub-matrix of B of order $(r+1)$.

Then $|B_0| = 0$ [if B_0 do not contains i^{th} row]

[$\because$ in this case B_0 is also a sub-matrix of A]

Again, if B_0 contains i^{th} row then we write

$|B_0| = k|A_0|$, where A_0 is a sub-matrix of A.

$\Rightarrow |B_0| = k|A_0| = 0$ $\qquad\qquad$ [$\because |A_0| = 0$]

Hence, $\rho(B) \le r = \rho(A)$

Again, if we apply $R_i \to \frac{1}{k} R_i$ in B, we obtained A and in this way we can show $\rho(A) \le \rho(B)$.

Thus $\rho(A) = \rho(B)$.

c) Addition of a row with some multiple of another row: Let us transform A by $R_i \to R_i + kR_j$.

Let B be the matrix obtained from A by this transformation.

Now, consider a sub-matrix B_0 of B of order $(r+1)$. If B_0 do not contains i^{th} row of B, then B_0 will also be a sub-matrix of A and hence $|B_0| = 0$.

Now, if B_0 contains the i^{th} row and j^{th} row of matrix B then $|B_0| = |A_0|$, where A_0 is corresponding placed sub-matrix of A.

Since, $|A_0| = 0$.

Therefore, $|B_0| = 0$.

If B_0 contains i^{th} row of B but does not contains j^{th} row of B.

Then, $|B_0| = |A_0| + k|C_0|$ where C_0 is a matrix of order $(r+1)$ formed with $(r+1)$ rows of matrix A.

Therefore, $|C_0| = 0$.

$\Rightarrow |B_0| = 0 + k \cdot 0$

Hence, every sub-matrix of B of order $(r+1)$ has determinant value zero.

$\Rightarrow \rho(B) \le r = \rho(A)$ ……………………. (a)

Now, if we apply the transformation $R_i \to R_i - kR_j$ on B, we get matrix A.

And in this way we can show $\rho(A) \le \rho(B)$ ………………….. (b)

From (a) and (b), we get

$$\rho(A) = \rho(B)$$

NOTE: By the similar method we can show that the elementary column transformations do not alter rank of a matrix.

8. The matrix from the system is

$$\begin{pmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \\ 5 \end{pmatrix}$$

Augmented matrix $C=[A:B]$ is

$$C=\begin{vmatrix} 5 & 3 & 7 & : & 4 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{vmatrix}$$

$$\sim\begin{vmatrix} 15 & 9 & 21 & : & 12 \\ 15 & 130 & 10 & : & 45 \\ 7 & 2 & 10 & : & 5 \end{vmatrix} \qquad \text{Applying } R_1 \to 3R_1 \text{ and } R_2 \to 5R_1$$

$$\sim\begin{vmatrix} 15 & 9 & 21 & : & 12 \\ 0 & 121 & -11 & : & 33 \\ 7 & 2 & 10 & : & 5 \end{vmatrix} \qquad \text{Applying } R_2 \to R_2 - R_1$$

$$\sim\begin{vmatrix} 35 & 21 & 49 & : & 28 \\ 0 & 11 & -1 & : & 3 \\ 35 & 10 & 50 & : & 25 \end{vmatrix} \qquad \text{Applying } R_1 \to \frac{7}{3}R_1, R_3 \to 5R_3 \text{ and } R_2 \to \frac{1}{11}R_2$$

$$\sim\begin{vmatrix} 35 & 21 & 49 & : & 28 \\ 0 & 11 & -1 & : & 3 \\ 0 & -11 & 1 & : & -3 \end{vmatrix} \qquad \text{Applying } R_3 \to R_3 - R_1$$

$$\sim\begin{vmatrix} 1 & \frac{3}{5} & \frac{7}{5} & : & \frac{4}{5} \\ 0 & 11 & -1 & : & 3 \\ 0 & 0 & 0 & : & 0 \end{vmatrix} \qquad \text{Applying } R_1 \to \frac{1}{35}R_1 \text{ and } R_3 \to R_3 + R_2$$

$$=[A_1 : B_1] \qquad \text{(say)}$$

$$\Rightarrow A \sim \begin{vmatrix} 1 & \frac{3}{5} & \frac{7}{5} \\ 0 & 11 & -1 \\ 0 & 0 & 0 \end{vmatrix} = A_1$$

Thus, $\rho(A) = 2 = \rho([A:B])$.

$\Rightarrow$ System is consistent.

And equivalent system is obtained by $[A_1 : B_1]$ i.e. $A_1X = B_1$.

$$\Rightarrow \begin{pmatrix} 1 & \frac{3}{5} & \frac{7}{5} \\ 0 & 11 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ 3 \\ 0 \end{pmatrix}$$

$5x + 3y + 7z = 4$ .... (i)

$11y - z = 3$ .... (ii)

We have, $n = 3$ and $r = \rho(A) = \rho([A:B]) = 2$.

$\therefore$ For solutions we have to fix $n - r = 3 - 2 = 1$ variable.

Let $z = k$

From (ii), $y = \frac{3+k}{11}$ and from (i), $x = \frac{4 - 3y - 7z}{5} = \frac{4 - 3\left(\frac{3+k}{11}\right) - 7k}{5}$

$\Rightarrow x = \frac{35 - 80k}{55}$, $y = \frac{3+k}{11}$ and $z = k$ are solutions, where k is arbitrary parameter.

9. The characteristic equation of the matrix A is $|A - \lambda I| = 0$

that is $\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\Rightarrow \lambda(\lambda - 3)(\lambda - 15) = 0$

Hence, eigen values are $0, 3, 15$.

Now, we obtain eigen vectors belonging to above eigen values.

Eigen vectors belonging to eigen value $\lambda = 0$ are the solutions of

$AX = \lambda X$ i.e. $AX = 0$ where $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

then $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\sim \begin{pmatrix} 0 & 10 & -10 \\ 0 & -5 & 5 \\ 2 & -4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ by $R_1 \to R_1 - 4R_3$ and $R_2 \to R_2 + 3R_3$

$\sim \begin{pmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ .... (i) by $R_3 \leftrightarrow R_1$ then by $R_3 \to R_3 + 2R_2$

Since rank of coefficient matrix $= 2$ and number of variables $= 3$.

Therefore, dimension of solution set of $(AX = 0) = 3 - 2 = 1$.

And, the reduced system is given by (i)

Also $2x - 4y + 3z = 0$ .... (ii)

$-5y + 5z = 0$ .... (iii)

Hence, there is one free variable, take $z = 1$

So we get from (ii) and (iii)

$x = \frac{1}{2}$ and $y = 1$.

$\therefore$ Solution space of (i) = set of eigen vectors belonging to $\lambda = 0$ has basis

$$B = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

and general element of this set = general eigen vector for $\lambda = 0 = k \begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix}$.

Eigen vectors belonging to eigen value $\lambda = 3$.

In this case eigen vectors are the solution of $AX = \lambda X$ for $\lambda = 3$.

or, $(A - 3I)X = 0$

or, $$\begin{pmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} -1 & -2 & -2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{by } R_1 \to R_1 + R_2$$

$$\sim \begin{pmatrix} -1 & -2 & -2 \\ 0 & 16 & 8 \\ 0 & -8 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{by } R_2 \to R_2 - 6R_1 \text{ and } R_3 \to R_3 + 2R_1$$

$$\sim \begin{pmatrix} -1 & -2 & -2 \\ 0 & 16 & 8 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{by } R_3 \to R_3 + \frac{1}{2} R_2$$

$\Rightarrow -x - 2y - 2z = 0$ .... (iv)

$16y + 8z = 0$ .... (v)

We have two equations and three variables i.e. $n=3$ and $r=2$.

$\Rightarrow$dimension of solution space $=n-r=3-2=1$.

For finding basis of solution space take $y=1$, we get from (iv) and (v)

$$z=-2,\ x=2$$

$\therefore$ Basis of solution space $=\left\{\begin{bmatrix}2\\1\\-2\end{bmatrix}\right\}$ and general solution of the system (eigen vectors) $=k\begin{bmatrix}2\\1\\-2\end{bmatrix}$, where k is arbitrary parameter.

Eigen vectors belonging to eigen value $\lambda=15$.

In this case eigen vectors are the solution of $AX=\lambda X$ for $\lambda=15$.

or, $(A-15I)X=0$

$$\text{or, } \begin{pmatrix}-7 & -6 & 2\\ -6 & -8 & -4\\ 2 & -4 & -12\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\sim \begin{pmatrix}-1 & 2 & 6\\ -6 & -8 & -4\\ 2 & -4 & -12\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \qquad \text{by } R_1\to R_1-R_2$$

$$\sim \begin{pmatrix}-1 & 2 & 6\\ 0 & -20 & -40\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \qquad \text{by } R_1\to R_2-6R_1 \text{ and } R_3\to R_3+2R_1$$

$$\Rightarrow -x+2y+6z=0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (vi)$$

$$-20y-40z=0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (vii)$$

We have two ($r=2$) equations and three ($n=3$) variables.

Thus, dimension of solution set $=n-r=1$ and in order to find its basis take $z=1$.

From equation (vi) and (vii), we get

$$y=-2, \; x=2$$

$\therefore$ Basis of solution set $=\left\{\begin{bmatrix}2\\-2\\1\end{bmatrix}\right\}$ and general solution (eigen vectors) $=k\begin{bmatrix}2\\-2\\1\end{bmatrix}$

, where k is arbitrary parameter.

10. Take $u,v\in R^2$.

then $u=(u_1,u_2)$ and $v=(v_1,v_2)$ where $u_1,u_2,v_1,v_2\in R$

Let $\alpha\in R=K$.

(a)
$$\begin{aligned}T(u+v)&=T\left((u_1,u_2)+(v_1,v_2)\right)\\&=T(u_1+v_1,u_2+v_2)\\&=\left(2(u_1+v_1)+3(u_2+v_2),4(u_1+v_1)-5(u_2+v_2)\right)\\&=\left(2u_1+3u_2+2v_1+3v_2,4u_1-5u_2+4v_1-5v_2\right)\\&=\left(2u_1+3u_2,4u_1-5u_2\right)+\left(2v_1+3v_2,4v_1-5v_2\right)\\&=T(u_1,u_2)+T(v_1,v_2)\\&=T(u)+T(v)\end{aligned}$$

(b)
$$\begin{aligned}T(\alpha u)&=T\left(\alpha(u_1,u_2)\right)\\&=T(\alpha u_1,\alpha u_2)\\&=\left(2\alpha u_1+3\alpha u_2,4\alpha u_1-5\alpha u_2\right)\\&=\alpha\left(2u_1+3u_2,4u_1-5u_2\right)\\&=\alpha T(u_1,u_2)\\&=\alpha T(u)\end{aligned}$$

Hence, T is a linear transformation (or linear operator).

(ii) Usual basis of $R^2 = \{(1,0),(0,1)\}$

$$= \{e_1, e_2\}$$

$$T(e_1) = T(1,0) = (2\cdot 1 + 3\cdot 0, 4\cdot 1 - 5\cdot 0)$$

$$= (2,4)$$

$$= 2(1,0) + 4(0,1)$$

$$= 2e_1 + 4e_2$$

$$\Rightarrow T(e_1) = 2e_1 + 4e_2$$

and $T(e_2) = T(0,1) = (2\cdot 0 + 3\cdot 1, 4\cdot 0 - 5\cdot 1)$

$$= (3,-5)$$

$$= 3(1,0) - 5(0,1)$$

$$= 3e_1 - 5e_2$$

$$\Rightarrow T(e_2) = 3e_1 - 5e_2$$

Hence, $T(e_1) = 2e_1 + 4e_2$ and $T(e_2) = 3e_1 - 5e_2$

$\Rightarrow$ Matrix of T relative to basis $\{e_1, e_2\} = [T] = \begin{pmatrix} 2 & 3 \\ 4 & -5 \end{pmatrix}$.

(iii) Let $v_1 = (1,2)$ and $v_2 = (2,5)$.

Then, $T(v_1) = T(1,2) = (2\cdot 1 + 3\cdot 2, 4\cdot 1 - 5\cdot 2) = (8,-6)$

Now we have to express $(8,-6)$ as a linear combination of v_1 and v_2.

Let $(8,-6) = \alpha(1,2) + \beta(2,5)$

$$= (\alpha + 2\beta, 2\alpha + 5\beta)$$

$\Rightarrow \alpha + 2\beta = 8$ ……………………….. (i)

$2\alpha + 5\beta = -6$ ……………………….. (ii)

On solving equation (i) and (ii), we get $\alpha = 52$ and $\beta = -22$.

Hence, $T(v_1) = 52(1,2) - 22(2,5)$

$$\Rightarrow T(v_1) = 52v_1 - 22v_2$$

Now, $T(v_2) = T(2,5) = (19,-17)$

Now we have to express $(19,-17)$ as a linear combination of v_1 and v_2.

Let $(19,-17) = \gamma(1,2) + \delta(2,5)$

$$= (\gamma + 2\delta, 2\gamma + 5\delta)$$

$\Rightarrow \gamma + 2\delta = 19$ ……………………….. (iii)

$2\gamma + 5\delta = -17$ ……………………….. (iv)

On solving equation (iii) and (iv), we get $\gamma = 129$ and $\delta = -55$.

Hence, $T(v_2) = 129(1,2) - 55(2,5)$

$$\Rightarrow T(v_2) = 129v_1 - 55v_2$$

$$\Rightarrow T(v_1) = 52v_1 - 22v_2$$

and $T(v_2) = 129v_1 - 55v_2$

Therefore, matrix of T relative to basis $S = \{v_1, v_2\} = \{(1,2),(2,5)\}$ is

$$[T] = [T]_S = \begin{pmatrix} 52 & 129 \\ -22 & -55 \end{pmatrix}.$$